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M[any] vacua of IIB

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Abstract: Description of the spectrum of fluctuations around a commutative vacuum solution, as well as around a solution with degenerate commutator in IIB matrix model is given in terms of supersymmetric-Yang-Mills (YM) model. We construct explicitly the map from hermitean matrices to YM fields and study the dependence of the spectrum and respective YM model on the symmetries of the solution. The gauge algebra of the YM model is shown to contain local reparameterisation algebra as well as Virasoro one.

Keywords: M(atrix) Theories, Space-Time Symmetries, Gauge Symmetry, Non-Commutative Geometry.

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Contents

1. Introduction

IKKT, or IIB matrix model [1], is a statistical mechanic model which was introduced as a non-perturbative regularisation of IIB superstring model in the Schild gauge [2]. It is the string analog of lattice gauge models and provides a tool for non-perturbative numerical study of the string theory (M-theory) [3]–[7].

The picture one has in the IKKT model is closely related to the Connes' approach to the non-commutative geometry [8]. In this approach a manifold is described in terms of the so-called Connes' triple $(\mathcal{A}, \Delta, \mathcal{H})$, rather than as a set of points. In this triple, $\mathcal{A}$ is the algebra of bounded operators (algebra of functions), Δ is some elliptic differential operator (Laplace or Dirac operator, for simplicity we consider euclidean signature) and $\mathcal{H}$ is the Hilbert space where $\mathcal{A}$ and Δ are represented. "Points" of such a manifold can be identified with the spectra of some position operators built from the triple, other local and global characteristics can as well be extracted from it.

In the case of IKKT matrix model the rôle of $\mathcal{A}$ is played by hermitean matrices with "smooth" eigenvalue distribution and bounded square trace in the limit $N \to \infty$, while the Δ and $\mathcal{H}$ are represented by the background solution and the adjoint representation of $\mathrm{U}(N)$.

Compactifications of this model to d-dimensional non-commutative tori was shown to result in non-commutative Yang-Mills (YM) models in respective dimensions [9]. In particular, compactification to a circle yields the Banks-Fischler-Shenker-Susskind (BFSS) matrix model which was introduced as a non-perturbative regularisation of the light-cone membrane action in $D = 11$, [10].

The compactification in ref. [9] is introduced as a restriction of the background solution to satisfy some periodicity conditions modulo gauge transformations. The respective periods are identified with the periods of the torus on which the model is compactified.

From the other hand, given a BPS background, i.e. a solution to equations of motion which preserves a part of the supersymmetry, one can map the space of $N \times N$ hermitean matrices of IKKT model to the space of real or matrix valued functions on some non-commutative manifold. Under certain conditions this map is isomorphism of algebras where the matrix product is mapped into non commutative or star product of functions due to the noncommutativity of the manifold. The properties of the manifold are exclusively determined by the respective BPS background solution.

Explicitly the BPS solution is given by a set of matrices having scalar commutators. In the case when the matrix of commutators is non degenerate (or for the subset on which it is non degenerate) one can easily construct the map from the space of arbitrary hermitean matrix fluctuations around respective BPS background to the space of functions on a non-commutative manifold [6]. The limit of commutative manifold corresponds to infinite commutator of the solution.

The case when the commutator is degenerate corresponds just to an opposite situation. One may argue, and this was the main reason for arriving at actual paper, that only commutative solutions give the global minimum of the IKKT actions. However, as we show in section 3, this is not exactly the case, since due to homogeneity of the IKKT action in bosonic fields, the respective action vanishes on purely bosonic solutions. Unfortunately, this regretful error committed in the pioneering work [1], was reproduced in the succeeding papers.[1] In spite of this disappointing discovery the study of the spectrum of fluctuations around a commutative solution still presents some interest, in particular due to the fact that, as it will be seen below, the degeneracy in the commutator of the solution corresponds to increasing the dimensionality of corresponding non-commutative YM model, which gives a "physically" different model (compare this with the situation with the first and second class constraints, [12]).

To obtain a non-commutative U(1) YM model from the matrix fluctuations one has to impose some irreducibility condition, while allowing certain degeneracy to the background solution one comes to description of a "vector fibre bundle" over the non-commutative manifold which leads to "non-abelian" non-commutative YM. Respective degeneracy can be interpreted as compactification on many coinciding branes, while the non-degenerate case corresponds to a single brane [1].

The objective of the actual paper is to analyse the relation between the background commutative/degenerate vacuum solution and the properties of the resulting YM model. Although, the commutative solution is a particular case of a generic BPS

[1] See e.g. most recent paper [11].

solution, as we already mentioned, it is “physically” different, since it corresponds to a singular limit of the generic case.

The plan of the paper is as follows. In the next section we give a brief account of IIB matrix model. After that we consider a BPS solution, show that it corresponds to a vanishing action and consider in more details the commutative case. We impose a set of conditions such a solution must respect, and build explicitly the map between the matrix fluctuations and functions of non-commutative manifold which appears to be a “non-commutative” product of two commutative manifolds dual to each other. This allows to find in the section 5. a representation of IIB matrix model in terms of YM fields for both commutative and generic degenerate case.

Finally, we discuss the results and consider the consequences and possible generalisations of the actual analysis.

2. The IIB matrix model

The IKKT, or IIB matrix model, is described by the classical action:

$$S = -\frac{1}{g^2}\,\mathrm{tr}\left(\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu,\psi]\right), \tag{2.1}$$

where A_μ, $\mu = 1\ldots D = 10$, ψ and $\bar{\psi}$ are $N \times N$, $N \to \infty$ hermitean matrices which act on N-dimensional Hilbert space $\mathcal{H}$, with scalar product $\eta^\dagger\eta$, $\eta \in \mathcal{H}$. Spinor matrices $\bar{\psi}$ and ψ also carry the SO(10) Majorana-Weyl spinor index, while bosonic ones A_μ carry vector one. We assume summation convention for repeated indices.

Equations of motion corresponding to the action (2.1) look as follows:

$$[A_\mu,[A_\mu,A_\nu]] - \left[\bar{\psi},\Gamma_\mu\psi\right] = 0\,, \tag{2.2}$$

$$\Gamma^\mu[A_\mu,\psi] = \left[\bar{\psi},A_\mu\right]\Gamma^\mu = 0\,. \tag{2.3}$$

This model possesses an $\mathcal{N} = 2$ supersymmetry corresponding to the transformations,

$$\delta_{(1)}A^\mu = i\,\bar{\epsilon}\,\Gamma^\mu\psi\,, \tag{2.4}$$

$$\delta_{(1)}\psi = \frac{i}{2}[A_\mu,A_\nu]\Gamma^{\mu\nu}\epsilon \tag{2.5}$$

and

$$\delta_{(2)}A_\mu = 0\,, \tag{2.6}$$

$$\delta_{(2)}\psi = \xi\,, \tag{2.7}$$

where ϵ and ξ are Majorana-Weyl spinors, as well as a SU(N) gauge symmetry,

$$A_\nu \longrightarrow \mathrm{U}^{-1}\,A_\nu U\,, \tag{2.8}$$

$$\psi \longrightarrow U^{-1}\psi U\,, \tag{2.9}$$

$$\bar{\psi} \longrightarrow U^{-1}\bar{\psi}U \tag{2.10}$$

and U is $N \times N$ unitary matrix. Since only the adjoint representation is in use this symmetry is $\mathrm{SU}(N)/\mathbb{Z}_N$ rather than $\mathrm{U}(N)$. This will not remain true if "matter" fields are included, in this case $\mathrm{U}(1)$ and $\mathbb{Z}_N$ will act nontrivially on the fields in the fundamental representation of $\mathrm{U}(N)$

Another symmetry is already mentioned SO(10) "covariance" of 10-dimensional vectors and spinors.

3. The vacuum

In the limit $N \to \infty$ eqs. (2.2) and (2.3) have an important class of purely bosonic ($\psi = 0$) solutions which preserve a part of the supersymmetry (BPS solutions). These solutions are given by $A_\mu = p_\mu$, where matrices p_μ satisfy,

$$[p_\mu, p_\nu] = iB_{\mu\nu} \tag{3.1}$$

with $B_{\mu\nu}$ as $\mathrm{u}(N)$ matrix proportional to the unity one, $B_{\mu\nu} \sim \mathbb{I}$. For finite N the commutator (3.1) cannot be satisfied but only mod some matrix vanishing in the sense of operator norm as $N \to \infty$. A solution of the described type can be realised e.g. by shift-and-clock operators on functions defined on a lattice [9, 6].

Naively, action (2.1) computed on the solution (3.1) should be equal to [1],

$$S_{\mathrm{BPS}} = \frac{N}{4g^2}(B_{\mu\nu})^2 . \tag{3.2}$$

As one can see this expression does not realise the minimum (even local one) unless $B_{\mu\nu} = 0$, since one can devaluate the action (3.2) by a smooth variation of the solution, e.g. $\delta p_\mu = \lambda p_\mu$, which decreases the action if $\lambda < 0$. The problem is accomplished by the fact that the variation of the action is *first order* in λ, which should not be the case if one varies a background solution. Indeed, this problem is solved if we see that for any finite N the action computed on a purely bosonic solution of equations of motion vanishes. Thus, for any finite N the bosonic part of the action (2.1) can be rewritten as,

$$S = \frac{1}{4g^2}\operatorname{tr} A_\nu[A_\mu, [A_\mu, A_\nu]] \tag{3.3}$$

which vanishes identically if A_μ is a solution to (2.2) with zero ψ.

The case with infinite N, however, is still indefinite since it depends how the limit $N \to \infty$ is achieved. One can obtain the limit either from $p_\mu(N)$ satisfying equations of motion or from ones not satisfying them, in the last case the value of action depends on how fast $p_\mu(N)$ approaches a solution as N goes to infinity. Thus, for a configuration $p_\mu(N)$, satisfying,

$$[p_\mu, p_\nu] = iB_{\mu\nu}\mathbb{I}(N) , \tag{3.4}$$

where $\mathbb{I}(N) = \mathbb{I} + \epsilon(N)$ is a unity matrix approaching sequence. In order to approach (3.1), one has to require,

$$\operatorname{tr}\epsilon(N) = -N\,, \tag{3.5}$$

$$\lim_{N\to\infty} \epsilon(N) = 0\,, \tag{3.6}$$

where the limit is computed using the operator norm, $\|\epsilon\| = \sup_{\eta^\dagger\eta=1}\sqrt{\eta^\dagger\epsilon^\dagger\epsilon\eta}$, $\eta \in \mathcal{H}$. Equations (3.5) and (3.6) are contradictory unless we restrict the space of vectors η on which the operator norm is computed e.g. by vectors having a fixed finite number of non-zero coordinates in the limit $N \to \infty$. In this case, the action (2.1) computed on this sequence of configurations reads,

$$S_{\mathrm{BPS}}(N) = \frac{1}{4g^2}B^2\operatorname{tr}\mathbb{I}(N)^2 = \frac{1}{4g^2}B^2\left(\operatorname{tr}\epsilon(N)^2 - N\right), \tag{3.7}$$

where the factor $(\operatorname{tr}\epsilon(N)^2 - N)$ is ambiguous. The additional requirement that for all N the configuration p_μ to be a solution to (2.2) eliminates the ambiguity since for any N one has $S_{\mathrm{BPS}}(N) = 0$, so we assume this requirement to be satisfied.

The vanishing of the classical BPS action results in "equality" in the factor e^{-S} between solutions p_μ with different commutators (3.1), either BPS or non BPS. The BPS vacua are, however, preferred since one expects to have no loop corrections to them [1].

The above problems are absent in the case of commutative solutions, i.e. when $B_{\mu\nu} = 0$. Before considering this case in more details, consider a generic solution (3.1). By a proper linear transformation $B_{\mu\nu}$ can be brought to the "standard" form having block diagonal form with two-dimensional blocks of the form,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and an $r \times r$ zero block, corresponding to zero modes of $B_{\mu\nu}$.

The set of matrices (p_μ) is split in this case in three subsets (p_i, q^i, p_a), where p_i and q^i form canonical conjugate pairs, and $p_a = z^\mu_a\, p_\mu$, $a = 1, \ldots, r = \operatorname{corank} B$, correspond to zero modes z^μ_a of $B_{\mu\nu}$, $B_{\mu\nu} z^\nu_a = 0$,

$$\left[p_i, q^j\right] = -i\,\delta^j_i\,, \tag{3.8}$$

$$[p, p] = [q, q] = 0\,. \tag{3.9}$$

Intuitively, at this stage one can see that p_μ corresponding to a non-degenerate part of $B_{\mu\nu}$ carries 1 (one-particle hamiltonian) degree of freedom while one corresponding to zero mode caries 2 degrees. The second degree of freedom should be given by the canonical conjugate q^a, if it exists. We hope this statement will become more transparent below.

For a non-zero $B_{\mu\nu}$, as well as for zero one it can be seen that the set of adjoint operators $P_\mu = [p_\mu, \cdot]$, (which act on $\mathrm{u}(N)$ matrices), is a commutative one,

$$(P_\mu P_\nu - P_\nu P_\mu)a = [p_\mu, [p_\nu, a]] - [p_\nu, [p_\mu, a]] = i[B_{\mu\nu}, a] = 0\,. \tag{3.10}$$

This fact common with one that P_μ are selfadjoint with respect to the scalar product,

$$(a, b) = \mathrm{tr}\, a^\dagger b\,, \tag{3.11}$$

where $a^\dagger$ stands for the hermitean conjugate matrix, means that P_μ can be diagonalised even for non-commutative set of matrices p_μ satisfying eq. (3.1). One can, therefore, decompose any hermitean $N \times N$ matrix in the orthogonal basis of P_μ eigenmatrices. Having this decomposition at hand we will construct in the next section the map from the space of hermitean matrices to the space of real functions on some non-commutative manifold.

In the case when $B_{\mu\nu} = 0$ one can diagonalise not only adjoint operators P_μ but also all matrices p_μ.

We say that p_μ is a non-degenerate commutative vacuum background solution if the following are satisfied:

- All p_μ are commutative,
$$[p_\mu, p_\nu] = 0\,. \tag{3.12}$$
- Matrices p_μ are (functionally) independent, this implies also linear independence,
$$\alpha^\mu\, p_\mu = 0 \Longrightarrow \text{all } \alpha^\mu = 0\,. \tag{3.13}$$
- They satisfy,
$$\mathrm{tr}\, p_\mu = 0\,, \tag{3.14}$$
$$\mathrm{tr}\, p_\mu\, p_\nu = 0 \quad \text{for } \mu \neq \nu\,. \tag{3.15}$$
- Eigenvalues of p_μ form a D-dimensional irregular lattice with sites symmetrically distributed with respect to the origin (the centre of the lattice), in the range $|\lambda| \leq \Lambda$,[2] where $|\lambda| = \sqrt{\lambda_\mu^2}$. In the limit $N \to \infty$ the lattice becomes dense.

  This implies that one can rearrange the matrix labels (which is equivalent to a gauge transformation) in such a way that eigenvalues λ_μ are distributed according to "smooth" and "monotonic" functions of the matrix index s_ν, $\lambda_\mu(s_\nu)$,

[2] One can alternatively chose eigenvalues to lie in other compact domain, e.g. a box, $\Lambda_\mu \leq \lambda_\mu \leq \Lambda_\mu$.

which is the s_ν-th eigenvalue, where s_ν form a linear D-dimensional lattice,

$$\det\left(\frac{\partial\lambda_\mu(s)}{\partial s_\nu}\right) > 0\,, \tag{3.16}$$

$$\lambda_\mu(-s) = -\lambda_\mu(s)\,, \tag{3.17}$$

$$|\lambda_\mu(s+1) - \lambda_\mu(s)| = \left|\frac{\partial\lambda_\mu(s)}{\partial s_\nu}\right| = o(N^{-1}), \quad N\to\infty, \Lambda = \text{fixed}\,. \tag{3.18}$$

The last gives the smooth distribution of eigenvalues in the limit $N\to\infty$. In this limit quantity Λ plays the role of UV cutoff. This implies that for a finite but large values of N one is justified to manipulate with $\lambda_\mu(s)$ as with quasicontinuous quantities.

The authentic continuum limit is achieved by sending Λ to infinity after the limit $N\to\infty$ is reached.

The eigenvalue problem in the case of commuting p_μ is equivalent to the Cartan decomposition problem in Lie algebra $\mathrm{u}(N)$, see e.g. [13]. In this context the above enlisted conditions say that p_μ must form the orthogonal basis in a D-dimensional plane of Cartan subalgebra of $\mathrm{su}(N)$ to which *no* root is orthogonal.

4. The map

As usual any matrix a is a function of a pair of labels (s,s'), $a = a(s,s')$. Matrix product and trace are given by, respectively,

$$(a\cdot b)(s',s'') = \sum_{\{s_\mu\}} a(s',s)b(s,s'')\,, \tag{4.1}$$

$$\mathrm{tr}\, a = \sum_{\{s_\mu\}} a(s,s)\,. \tag{4.2}$$

In what follows we will use also quasicontinuous notations where the sums we will substitute by the integrals. Taking into account that the increment of s in (4.1) and (4.2) is $ds = \Delta s = 1$ these sums can be written as following integrals,

$$(a\cdot b)(s',s'') = \int ds\, a(s',s)b(s,s'')\,, \tag{4.3}$$

$$\mathrm{tr}\, a = \int ds\, a(s,s)\,. \tag{4.4}$$

Where the range of integration is given by $-N_\mu/2 \le s_\mu \le N_\mu/2$ and $\prod_\mu N_\mu = N$.

The unity matrix is given by the δ symbol,

$$\mathbb{I}(s,s') = \delta_{ss'} \equiv \delta(s-s')\,. \tag{4.5}$$

Note, that in these notations $\delta(0) = 1$, this differs from the usual "continuous" δ-function by a diverging factor of order N.

The limit $N \to \infty$ is achieved in the following way. First, one introduce a UV cutoff L and rescale $s \to \frac{2L}{N}s$. Thus the increment Δs becomes really small and one has the genuine integration in eqs. (4.1)–(4.4). The cutoff removing is obtained in the limit $L \to \infty$. Since we plan to identify p_μ with momentum operators the continuous λ_μ spectrum corresponds to non-compact manifolds, while to have the compact result one has to keep the discreteness of the spectrum and sums instead of the integrals.

In this picture matrices p_μ look as follows:

$$p_\mu(s, s') = \lambda_\mu(s)\delta(s - s')\,, \tag{4.6}$$

where $\lambda_\mu(s)$ scan the eigenvalue lattice of p_μ, while an arbitrary diagonal matrix ϖ looks like,

$$\varpi(s, s') = \varpi(s)\delta(s - s')\,. \tag{4.7}$$

It is not difficult to compute the action of operator P_μ on an arbitrary matrix $a(s, s')$,

$$\begin{aligned}(P_\mu a)(s, s') &= [p_\mu, a](s, s') = \\ &= \int ds''(\lambda_\mu(s)\delta(s - s'')a(s'', s') - a(s, s'')\lambda_\mu(s'')\delta(s'' - s')) = \\ &= (\lambda_\mu(s) - \lambda_\mu(s'))a(s, s')\,.\end{aligned} \tag{4.8}$$

One immediately finds that the spectrum of P_μ is given by eigenvalues $k_\mu = \lambda_\mu(t) - \lambda_\mu(t')$ for any t and t' taking values in the lattice $\{s_\mu\}$, and corresponding to eigenvectors,

$$f_{tt'}(s, s') = \delta(s - t)\delta(s' - t')\,, \tag{4.9}$$

$$P_\mu f_{tt'} = (\lambda_\mu(t) - \lambda_\mu(t'))\, f_{tt'}\,. \tag{4.10}$$

For large values of N eigenvalues k become degenerate, the respective eigenfunctions can be labelled by eigenvalue k_μ and a label t counting the degenerate states,

$$f_t(k) = \int dt'\delta(t' - s(\lambda(s) - k))f_{tt'} = f_{ts(\lambda(s)-k)}\,, \tag{4.11}$$

where the function $s(\lambda)$ is the inverse to the $\lambda(s)$: $s(\lambda(s)) = s$, $\lambda(s(\lambda)) = \lambda$.

Now, consider the matrix q^μ defined by,

$$q^\mu = -i\frac{\partial}{\partial k_\mu}\int dt f_t(k)\Big|_{k=0} = iA_\alpha{}^\mu(s)\delta'^\alpha(s' - s)\,, \tag{4.12}$$

where

$$A_\alpha{}^\mu(s) = \left.\frac{\partial s_\alpha}{\partial \lambda_\mu}\right|_{\lambda=\lambda(s)} = \left(\left(\frac{\partial \lambda}{\partial s}\right)^{-1}(s)\right)_\alpha{}^\mu\,, \tag{4.13}$$

and $\delta'^\alpha(s) = \frac{\partial}{\partial s_\alpha}\delta(s)$.

As one can immediately see the commutator of p_ν with q^μ is the canonical one,

$$[p_\mu, q^\nu] = -i\,\delta_\mu^\nu \mathbb{I}\,, \tag{4.14}$$

i.e. p_μ and q^ν form a canonical conjugate pair and can be interpreted as, respectively, momentum and coordinate operators, as we anticipated earlier. Strictly speaking, such pairs of operators exist only in the limit $N \to \infty$, i.e. when the spectrum of either q or p is continuous.

From q's one can construct a matrix $E(k) = e^{ik_\mu q^\mu}$. Matrix $E(k)$ is non degenerate and it is an eigenvector of P_μ corresponding to the value k,

$$[p_\mu, E(k)] = k_\mu E(k)\,, \tag{4.15}$$

having the squared norm,

$$\|E(k)\|^2 = \mathrm{tr}\, E^\dagger(k)E(k) = N\,. \tag{4.16}$$

For a finite N the dimensionality of k-eigenspace depends on k and decrease as k approaches to 2Λ since matrix edges are approaching for large values of k. When $N \to \infty$ this difference disappears and one can regard the spaces with different k as isomorphic. This approximation is accurate as soon as $|t - t'| \ll N$, where $\lambda(t) - \lambda(t') = k$. Matrix $E(k)$ can be considered as the eigenvalue shift operator,

$$E(k)f(k') \sim f(k + k')\,. \tag{4.17}$$

Due to isomorphism and nondegeneracy of $E(k)$, eigenvectors corresponding to a non-zero k are given by the product of zero vectors and matrix $E(k)$ as follows:

$$E_t(k) = f_t(0)E(k)\,, \tag{4.18}$$

where $f_t(0)$ is the basis in the space of diagonal matrices which form the zero space. It is transparent that this basis is orthogonal in the space of $N \times N$ if $f_t(0)$ form an orthonormal basis in the space of diagonal matrices, the norm of $E_t(k)$ coinciding with that of $f_t(0)$. Thus, given an orthonormal basis in the zero space of P_μ one can spread it out through all k-eigenspaces using eq. (4.18).

Since eigenvectors of P_μ form a complete orthonormal set in the space of $N \times N$ matrices, any hermitean matrix a can be expanded in the basis of P_μ eigenvectors,

$$a = \sum_{t,k} \tilde{a}_t(k) f_t(k) = \sum_{t,k} \tilde{a}_t(k) f_t(0) e^{ik\cdot q}\,. \tag{4.19}$$

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Orthonormality of the spectrum assures the inverse transformation,

$$\tilde{a}_t(k) = \frac{1}{\|f_t(0)\|^2}\,\mathrm{tr}\, e^{-ikq} f_t^*(0) a\,. \tag{4.20}$$

From the properties a vacuum solution one can trace that arbitrary matrix commuting with p_μ is a diagonal matrix (4.7), and it may be represented as a function of p_μ,

$$\varpi = \varpi(p) \equiv \varpi(s(p))\,, \tag{4.21}$$

where $\varpi(s(\lambda))|_{\lambda=p}$ is the same function as in (4.7), but computed on diagonal matrices p_μ.

One may expand ϖ in multiple ways e.g. in Fourier series as function of p_μ,

$$\varpi = \sum_z \tilde{\varpi}(z) e^{ip\cdot z}\,, \tag{4.22}$$

where z_μ are points of the lattice of eigenvalues of operator $Q^\mu = [q^\mu, \cdot]$,

$$\left[q^\mu, e^{ipz}\right] = -z^\mu e^{ipz}\,. \tag{4.23}$$

Since operators Q^μ are also commutative selfadjoint operators (they commute also with P_μ) their spectrum satisfy,

$$\mathrm{tr}\, e^{ip(z-z')} = N\delta_{zz'} \tag{4.24}$$

which gives the ground for inverse transformation,

$$\tilde{\varpi}(z) = \frac{1}{N}\,\mathrm{tr}\, e^{-ipz}\varpi\,. \tag{4.25}$$

Using the above decomposition for the diagonal matrices one can rewrite the expansion eq. (4.19) for arbitrary hermitean matrix a as follows:

$$a = \sum_{z,k} \tilde{a}(z,k) e^{-ikz} e^{ipz} e^{ikq} = \sum_{z,k} \tilde{a}(z,k) e^{ipz+ikq}\,, \tag{4.26}$$

$$\tilde{a}(k,z) = \frac{1}{N}\,\mathrm{tr}\, a e^{-ipz-ikq}\,, \tag{4.27}$$

$$\tilde{a}^*(k,z) = \tilde{a}(-k,-z)\,. \tag{4.28}$$

At this point one can identify the space of hermitean matrices with the space of real functions on the “non-commutative manifold” Spec $(P) \times$ Spec (Q), where Spec (P) and Spec (Q) are the varieties of eigenvalues of P, and respectively of Q. In fact when $N \to \infty$ with Λ fixed Spec (P) form a compact manifold, while Spec (Q) tends to a non-compact lattice. Continuum limit for this lattice is achieved in the limit $\Lambda \to \infty$.

This map is given by,[3]

$$a \longmapsto a(x,l) = \sum_{z,k} \tilde{a}(z,k) e^{ilz+ikx}\,, \tag{4.29}$$

where $\tilde{a}(z,k)$ is defined by eq. (4.27).

This map can be turned backward, therefore, it is one-to-one. The corresponding matrix is given by,

$$a = \sum_{k,z} \tilde{a}(k,z) e^{ipz+ikq}\,, \tag{4.30}$$

now, $\tilde{a}(z,k)$ is the Fourier transform of the function $a(x,l)$,

$$\tilde{a}(z,k) = \frac{1}{N} \sum_{x,l} a(x,l) e^{-ilz-ikx}\,. \tag{4.31}$$

Under the correspondence (4.29)–(4.31) the matrix product is mapped into the star product given by,

$$(a\cdot b) \longmapsto a \star b(x,l) = e^{-\frac{i}{2}\left(\frac{\partial^2}{\partial x' \partial l} - \frac{\partial^2}{\partial x \partial l'}\right)} a(x,l) b\left(x',l'\right)\Big|_{\substack{x'=x\\ l'=l}}\,, \tag{4.32}$$

while the trace corresponds to the lattice integration over x and l,

$$\operatorname{tr} a \longrightarrow \sum_{\substack{l\in \mathrm{Spec}(P)\\ x\in \mathrm{Spec}(Q)}} a(x,l) \equiv \int dx\, dl\, a(x,l)\,. \tag{4.33}$$

Commutators with p_μ and with q^μ give the lattice analogs of differentiation with respect to x^μ and respectively l_μ,

$$[p_\mu, a] \longrightarrow -i \frac{\partial}{\partial x^\mu} a(x,l)\,, \tag{4.34}$$

$$[q^\mu, a] \longrightarrow -i \frac{\partial}{\partial l_\mu} a(x,l)\,. \tag{4.35}$$

5. The spectrum

Let us now return back to the Matrix Model action (2.1) and consider arbitrary fluctuations around the vacuum solution p_μ satisfying (3.12)–(3.15),

$$A_\mu = p_\mu + g a_\mu\,, \tag{5.1}$$

here a_μ is an arbitrary hermitean matrix.

[3] We use the same character to denote both the matrix and corresponding function the difference being that for functions we write explicitly its arguments, i.e. if a denote some matrix we write $a(z,l)$ for the corresponding function and vice versa.

Perturbed action looks as follows:

$$S = -\operatorname{tr}\left(\frac{1}{4}([p_\mu, a_\nu] - [p_\nu, a_\mu] + g[a_\mu, a_\nu])^2 + \bar{\psi}\,\Gamma^\mu[(p_\mu + g a_\mu), \psi]\right). \tag{5.2}$$

Now, let us use the correspondence between matrices and functions to map the fluctuation a_μ to function $a_\mu(x, l)$. Then, action (5.2) is rewritten as follows:

$$S = -\int d^D x\, d^D l \left(\frac{1}{4}\mathcal{F}^2_{\mu\nu}(x, l) + \bar{\psi} \star \Gamma^\mu \nabla_\mu \psi(x, l)\right), \tag{5.3}$$

where $d^D x$ and $d^D l$ are invariant measures on Spec$\,(Q)$ and Spec$\,(P)$, respectively. They are given by eigenvalue distribution densities for P and Q. Also,

$$\mathcal{F}_{\mu\nu} = i\,\frac{\partial}{\partial x^\mu}\, a_\nu(x, l) - i\,\frac{\partial}{\partial x^\nu} a_\mu(x, l) - g[a_\mu, a_\nu]_\star(x, l)\,, \tag{5.4}$$

$$\nabla_\mu \psi(x, l) = i\,\frac{\partial}{\partial x^\mu}\, \psi(x, l) - g[a_\mu, \psi]_\star(x, l)\,, \tag{5.5}$$

where $[\cdot, \cdot]_\star$ stands for the star commutator defined as,

$$[a, b]_\star = a \star b - b \star a\,. \tag{5.6}$$

Action (5.3) give the *exact* description of IIB matrix model in $N \to \infty$ limit in terms of functions on the manifold Spec$\,(P) \times$ Spec$\,(Q)$. This action possesses a huge gauge symmetry given by the following transformations by a star-unitary function $U(x, l)$,

$$a_\mu(x, l) \longrightarrow U^{-1} \star a_\mu \star U(x, l) - \frac{i}{g}\, U^{-1} \star \frac{\partial}{\partial x^\mu}\, U(x, l)\,, \tag{5.7}$$

$$\psi(x, l) \longrightarrow U^{-1} \star \psi(x, l)\,, \tag{5.8}$$

where $U^{-1}(x, l) = U^\star(x, l)$, and star conjugate function is given by the function corresponding to the hermitean conjugate matrix and in this particular case coincides with the complex conjugate function,

$$U^\star(x, l) = \left(U^\dagger\right)(x, l) = U^*(x, l)\,. \tag{5.9}$$

Global gauge group G can be identified with "constant" transformations U, satisfying $\partial_\mu U = 0$. This means that U may depend only on the momentum parameter l, i.e. $U = \mathrm{U}(l)$. At first sight it seems that the gauge group is U(1) group localised along Spec$\,(P)$, since gauge symmetry is realised by scalar functions and not matrix valued ones. Indeed, the "global" group is abelian,

$$U(l) \star U'(l) = U(l)U'(l) = U'(l) \star U(l)\,, \tag{5.10}$$

but this commutativity does not hold for local transformations.

Consider the algebra of infinitesimal gauge transformations $\mathfrak{G}$. This is algebra of real functions with star commutator,

$$[f,g]_\star = ih, \quad f,g,h \in \mathfrak{G}\,. \tag{5.11}$$

Natural basis for this algebra form following functions,

$$L_{k,z}(x,l) = e^{ilz+ikx}\,. \tag{5.12}$$

The commutator for generators $L_{k,z}$ looks as follows:

$$[L_{k,z}, L_{k',z'}]_\star = 2i\sin\frac{1}{2}(k'z - kz')L_{k+k',z+z'}\,. \tag{5.13}$$

Thus, one may interpret the model (5.3) as an ordinary (commutative) Yang-Mills model with the local algebra of gauge transformations (5.13).

It is worthwhile to note that although the model we got is defined in the terms the flat space, the gauge symmetry the model has contains the group of local x-reparameterisations as a subgroup, whose infinitesimal transformations are generated by, $L_\epsilon = \epsilon^\mu(x)l_\mu$,

$$[L_\epsilon, f(x)] = \epsilon^\mu(x)\partial_\mu f(x)\,, \tag{5.14}$$

but this is not all. It is not difficult to show that algebra (5.13) contains Virasoro subalgebra which may serve as an indication that action (5.3) also describe the string spectrum. Indeed, consider a functions $\theta(x,l)$ defined $\mathrm{mod}\,2\pi$ and its canonical conjugate $w(x,l)$,

$$[w,\theta]_\star = -i\,. \tag{5.15}$$

Then, generators $L_n = e^{in\theta}w$ satisfy,

$$L_n^\star = L_{-n}\,, \tag{5.16}$$

$$[L_n, L_m]_\star = (n-m)L_{n+m} \tag{5.17}$$

which is exactly the classical Virasoro algebra.

The above results can be readily extended to the case of a generic background commutator $B_{\mu\nu}$. In this case we assume the subset (p_i, p_a) to satisfy conditions (3.12)–(3.18). Operators q^i, and respective eigenfunctions are already known, one has only to find the counterparts to p_a. Repeating the derivations of the previous section for this particular case, one comes to the map from matrices to functions $f(x^\mu, l^a)$, with star product defined as,

$$(a\star b)(x,l) = e^{-\frac{i}{2}\tilde{C}^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x'^\nu} - \frac{i}{2}\left(\frac{\partial^2}{\partial x'^a \partial l_a} - \frac{\partial^2}{\partial x^a \partial l'_a}\right)} a(x,l)b(x',l')\,, \tag{5.18}$$

tensor $\tilde{C}^{\mu\nu}$ is defined by,

$$\tilde{C}^{\mu\nu}B_{\nu\alpha} = B_{\alpha\nu}\tilde{C}^{\nu\mu} = \Pi^\mu_\alpha\,, \tag{5.19}$$

where Π^μ_α is the projector to the space orthogonal to zero modes of $B_{\mu\nu}$.

The YM action (5.3) keeps in this case the same form except the integration is done over $d^D x\, d^r l$, and the star product is given by eq. (5.18). As we see, zero modes of $B_{\mu\nu}$ lead to extra integrations in (5.3) over dl, or extra gauge (and, respectively, physical) degrees of freedom.

6. Generalisations and conclusions

In analysing the spectrum of fluctuations around a vacuum solution of the classical action (2.1) we made several assumptions about the respective solutions. Namely, we required p_μ to form an independent and non-degenerate commutative set of matrices. This lead us to the action (5.3) for fields on the product of the manifold corresponding to the spectrum of coordinate operator and its dual given by the spectrum of momentum operator. This representation contains a broad symmetry including invariance with respect to local reparameterisations as well as $2D$ conformal symmetry, and can be interpreted as a Yang-Mills model with gauge algebra given by (5.13).

The above assumptions seem natural, but one may pose a question: what may happen if one gives up some of them?

Consider first the case when all conditions are respected except the number of independent matrices is not equal to D but is smaller. Let, in particular, matrices p_μ be expressed as linear combinations of the independent subset (p_α) $\alpha = 1, \ldots, p+1 < D$,

$$p_\mu = \xi_\mu^\alpha\, p_\alpha\,, \tag{6.1}$$

where rank $\|\xi_\mu^\alpha\| = p+1$.

In this case, one can find the position operators corresponding to independent p_α, and a generic hermitean matrix will be expandable in the basis of the matrices $e^{ip_\alpha z^\alpha + ik_\alpha q^\alpha}$. Therefore, after manipulations like in previous section, hermitean matrices are now represented by the real functions defined on the spectrum of the subset of *independent* p_α and q^α, $\alpha = 1, \ldots, p+1$.

As a result, one has action (5.2) mapped into a $(p+p)$-dimensional action which corresponds to the reduction of the action (5.3) of $(D+D)$-dimensional YM model to a $(p+p+2)$-dimensional plane given by ξ_μ^α.

One may describe the above situation as a localisation of IIB matrix model to a p-brane in contrast to $(D-1)$-brane we had initially in (5.3).

Consider now the opposite case, i.e. when all matrices p_μ together fail to possess non-degenerate set of eigenvalues. The last means that (i) there exist eigenvalue sets (λ_μ) whose eigenspace are degenerate and, therefore, (ii) there are matrices π commuting with all p_μ but not being functions of p_μ. It is not difficult to show that the set of these matrices generate at most central extended Lie algebra $\mathfrak{L}$,

$$[\pi_a, \pi_b] = ic_{ab} \cdot \mathbb{I} + f_{ab}^c \pi_c\,. \tag{6.2}$$

Thus, one is able to map the generic hermitean matrix a from $\mathrm{u}(N)$ algebra to a $\mathfrak{L}$-valued function $a(x,l)$. In particular, when $\mathfrak{L} = \mathrm{u}(n)$ this corresponds to a "non-abelian" generalisation of the action (5.3). The "physical" interpretation of this solution is the localisation of IIB matrix model to n copies of coinciding branes, [1].

All above says that the spectrum of the matrix model is in a strong dependence of the vacuum solution chosen. Due to the limit $N \to \infty$, the different background solutions p_μ lead to different continuum models (5.3). One may conjecture that the respective models may be classified by the rank r of $B_{\mu\nu}$ and intersection of stabiliser groups of each p_μ, or by the symmetry of the vacuum.

In this paper we considered the solution with the smallest possible group of symmetry giving the D-dimensional action. In fact, this configuration has the "largest" measure (entropy factor), or better to say "the moduli space" among the commutative solutions. The opposite extreme is given by the "solution" where all p_μ are proportional to unity matrix $p_\mu = \lambda_\mu \mathbb{I}$ which describes the $\mathrm{u}(N)$ YM model localised on a point, i.e. the original matrix model. The "moduli space" of such solution is parameterised by just D numbers $\lambda_1, \ldots, \lambda_D$.

Thus, in IIB matrix model one has a plenty of vacua. For a finite N these vacua are connected through the fluctuations corresponding to zero modes of the Hessian matrix $S_{\mu\nu} \equiv \frac{\partial^2 S}{\partial A_\mu \, \partial A_\nu}(p)$. As $N \to \infty$, we expect such fluctuations to fall out of the class of allowed functions, namely continuous L^2-integrable functions, and the vacua to become separated.

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